Gravitational Lensing – Basic Knowledge

This document is to summarize basic knowledge related to the experiment, including the two cases you will simulate – lensing by a point mass and lensing by a singular isothermal sphere (i.e. a galaxy).

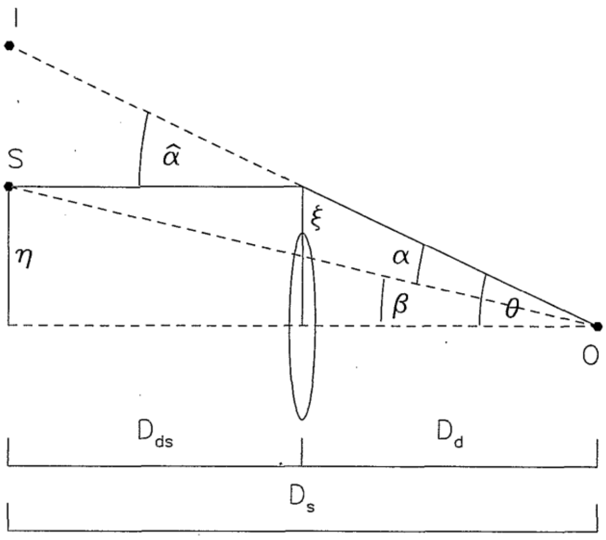

Figure (1): S is the source, η is the transverse distance from the optic axis, O is the observer, β and θ are the angular separations of the source and the image from the optic axis as seen by the observer. $\hat{\alpha}$ is the deflection angle.

Note: We will assume that S is a point source; more complex sources can be represented as a sum of multiple point sources.

Lensing by a point mass

A point mass may be represented by a lens with logarithmic shape

$$\Delta(\xi) = \Delta(\xi_0) + \frac{2R_{sc}}{(1-n)} \ln(\frac{\xi}{\xi_0}) \quad (1)$$

with $R_{sc} = \frac{4*G*M}{c^2}$ as shown in Fig (2). Here, n is the refractive index and M is the mass of the lensing point mass, and ξ_0 is the coordinate of the lens center.

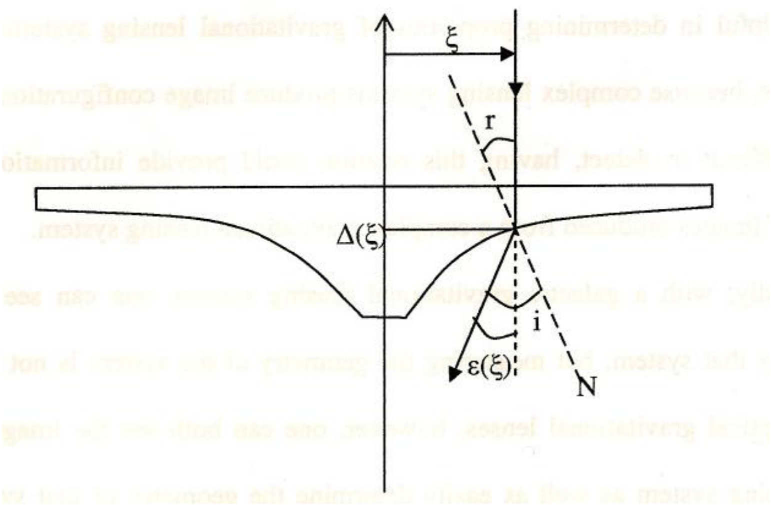

Figure (2): Logarithmic lens corresponding to a point mass.

(a) The Einstein ring

When the source S, center of the lens, and observer O are aligned along a straight line, the source S appears to the observer as a ring, known as the Einstein ring.

The radius of the Einstein ring satisfies the equation

$$\theta_E = \sqrt{\frac{R_{sc}}{D}} \tag{2}$$

where $D = \frac{D_d D_s}{D_{ds}}$, R_{sc} is as defined before, and D_d, D_s, D_{ds} are as shown in Fig (1).

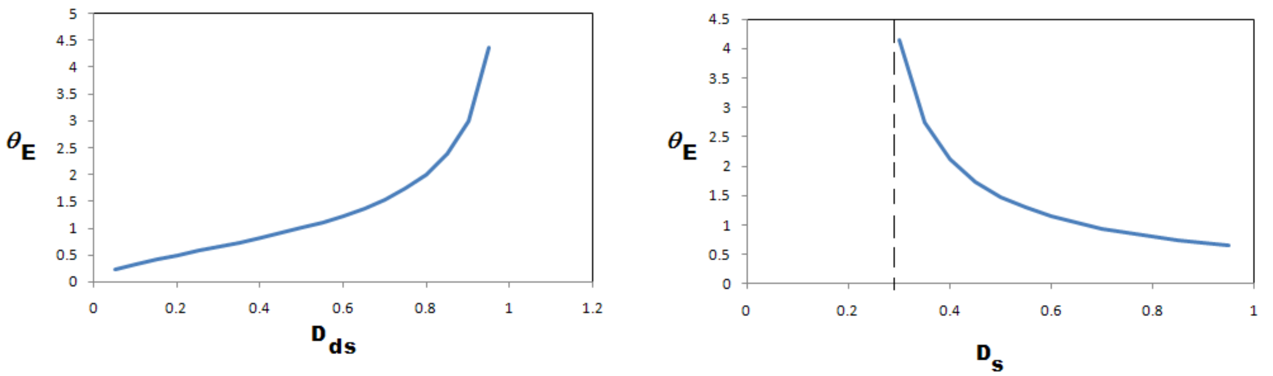

Figure (3): Functional dependences for the point mass lens. *(left):* Einstein ring radius vs. D_{ds} (D_s=1 fixed). *(right):* Einstein ring radius vs. D_s (D_{ds}=0.3 fixed).

Figure 3 depicts the functional dependence of the Einstein ring radius on various dimensional parameters. For a fixed D_s (source-observer distance) the Einstein ring radius increases as the lens approaches the observer ($D_{ds} \rightarrow D_s$ or equivalently $D_d \rightarrow 0$). For a fixed D_d or D_{ds}, the Einstein ring radius scales as the inversely to the source - observer distance D_s.

(b) Strong Lensing, double images

If the source is misaligned from the observer-lens axis by an angle β, the lens equation of this point-mass lens

$$\beta = \theta - \frac{\theta_E^2}{\theta} \tag{3}$$

has two solutions

$$\theta_\pm = \frac{1}{2}(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}) \tag{4}$$

Thus two images corresponding to the source are seen (the symmetry of the Einstein ring is broken by the source's misalignment).

The two images appear on either side of the source's actual position, with one image inside where the Einstein ring would be and the other outside (Fig. 4). As the source moves away from the lens-observer axis (i.e., as β increases), one of the images approaches the lens and becomes very faint, while the other image approaches closer and closer to the true position of the source and trends toward the source's actual brightness.

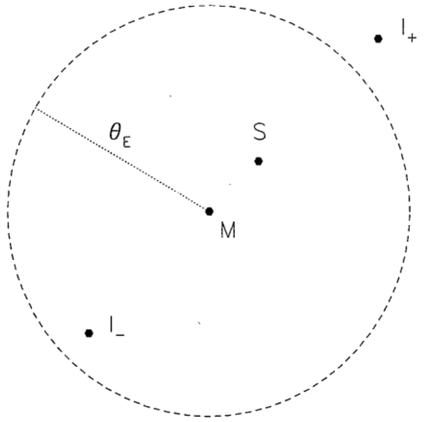

Figure (4): Relative locations of the source S and images I_+ and I_- lensed by a point mass *M*. The dashed circle is the Einstein ring with radius θ_E. One image is inside the Einstein ring and the other outside.

(c) Flux (Brightness) Amplification

For a circularly symmetric lens, the intensities of the images of the source S are amplified by a factor μ:

$$\mu = \frac{\theta}{\beta}\frac{d\theta}{d\beta} \tag{5}$$

For a point-mass lens the solutions are

$$\mu_{\pm} = \frac{u^2+2}{2u\sqrt{u^2+4}} \pm \frac{1}{2} \tag{6}$$

where $u = \beta\theta_E^{-1}$. The net amplification of flux in the two images is obtained by adding the absolute magnifications:

$$\mu = \frac{u^2+2}{2u\sqrt{u^2+4}} \tag{7}$$

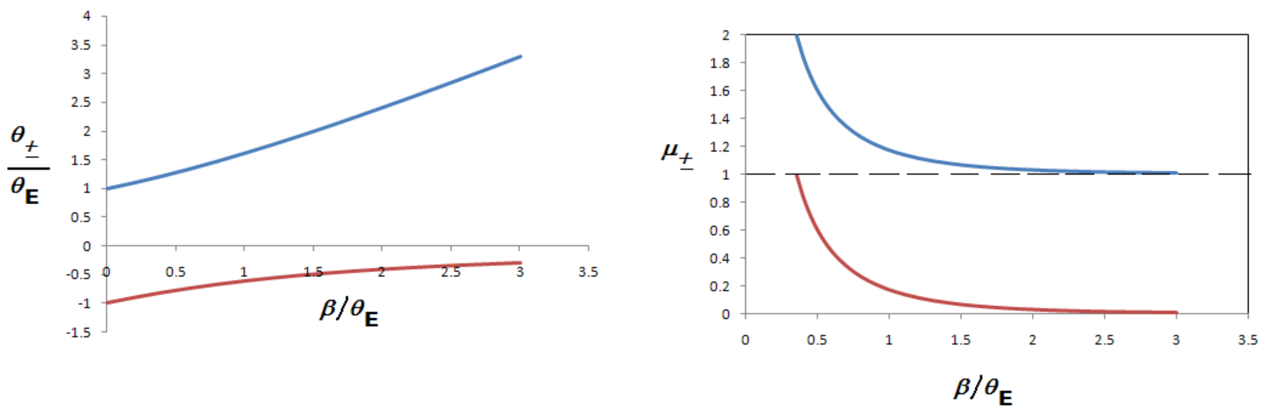

Figure (5): *(left)* Image locations θ_+ (blue) and θ_- (red) as a function of source offset angle β for point mass lensing. *(right)* Flux amplification factors μ_+ (blue) and μ_- (red) versus source offset.

Figure 5 shows the functional dependence of the image positions and amplifications versus the source offset angle β. For β=0, the images the light is distributed around a circle of radius θ_E (the Einstein ring); as β increases the symmetry is broken and the ring is replaced by two separate (and distorted) images. For β>0, the (+) image is outside the β=0 Einstein ring, and the (-) image is inside. For large β, the amplification factor of the (+) image tends toward unity and towards its true position in the sky, whereas the (-) image trends toward the source position as its intensity fades to zero.

Lensing by a singular isothermal sphere

The singular isothermal sphere is a simple model for galaxies, which have a mass distribution inversely proportional to r^2, corresponding to a conical lens shape

$$\Delta(\xi) = \Delta(\xi_0) + \frac{\hat{\alpha}}{(n-1)}(\xi_0 - \xi) \tag{8}$$

where $\hat{\alpha} = 4\pi \frac{\sigma_v^2}{c^2}$, σ_v is the one-dimensional velocity dispersion of the stars, n is the refractive index and ξ_0 is the coordinate of the lens center.

(a) The Einstein ring

When the source, center of the lens, and observer are aligned, as before the Einstein ring is observed. Its radius satisfies the equation

$$\theta_E = \hat{\alpha}\frac{D_{ds}}{D_s} \tag{9}$$

where $\hat{\alpha} = 4\pi \frac{\sigma_v^2}{c^2}$, and D_s and D_{ds} are as shown in Fig. (1).

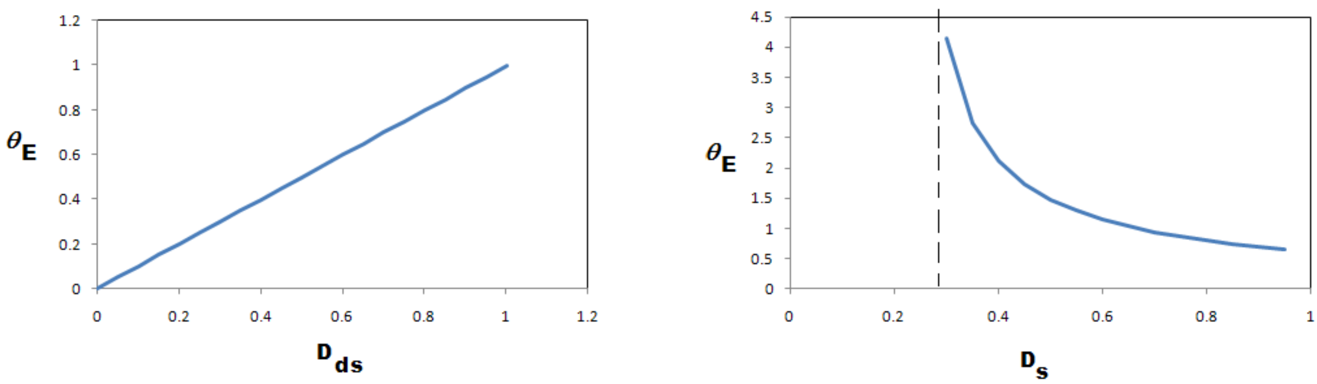

Figure (6): Functional dependences of the Einstein ring radius for the isothermal lens, as in Figure (3). Once again, as D_S increases, the Einstein ring becomes smaller, and as the source-lens distance grows (at constant D_S), so does the Einstein ring.

(b) Multiple Images

Multiple images are observed only if from the observer's position the source lies inside the Einstein ring, i.e. if $\beta < \theta_E$. When this condition is satisfied, the lens equation has two solutions

$$\theta_\pm = \theta_E \pm \beta \qquad (10)$$

and thus two images at the above positions are seen when $\beta \neq 0$.

If, however, the source lies outside the Einstein ring, i.e. if $\beta > \theta_E$, there is only one image seen, at

$$\theta = \theta_+ = \beta + \theta_E \qquad (11)$$

(d) Flux Amplification

The intensity amplification ratios of the two images may be found as

$$\mu_\pm = \frac{\theta_\pm}{\beta} = \frac{\theta_E}{\beta} \pm 1 \qquad (12)$$

where the μ_- solution only exists if $\beta < \theta_E$.

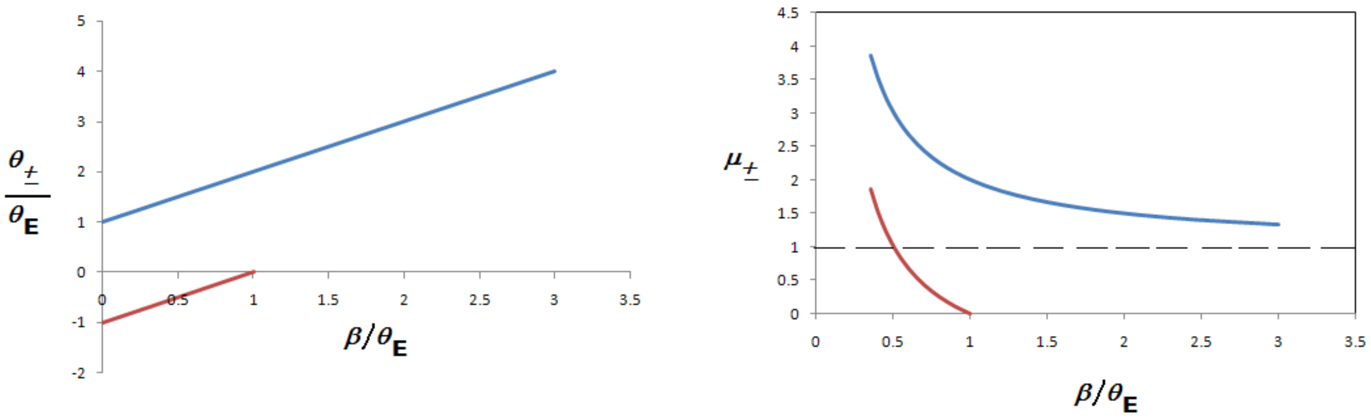

Figure (7): (left) Image locations θ_+ (blue) and θ_- (red) as a function of source offset angle β for the isothermal lens. (right) Flux amplification factors μ_+ (blue) and μ_- (red) versus source offset. Note that the (-) solution does not exist for $\beta/\theta_E>1$, and that the (+) solution trends toward the actual source position and intensity for $\beta>>\theta_E$.