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QUANTUM HALL EFFECT: THEORETICAL BACKGROUND

DAVID TAM
COLUMBIA UNIVERSITY DEPARTMENT OF PHYSICS
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1. INTRODUCTION

In this brief summary of the Integer Quantum Hall Effect (IQHE), I describe a system that, under the right conditions, exhibits astonishingly precise quantization of the quantity h/e^2 , both fundamental constants of nature¹. Our system is a two-dimensional sheet of electrons, which we manufacture at the junction between two semiconducting materials when cooled to the boiling point of liquid Helium (about 4 Kelvin). As we shall see, the electrons in the sheet can only possess energy of discrete values, called Landau energy levels, and because of the exclusion principle, only a certain number of electrons per sample can belong to each level. By applying a magnetic field, we change the number of electrons that fit; as electrons are reshuffled, we may observe the quantization by measuring the electrical properties of the sheet.

2. CLASSICAL HALL EFFECT

Consider a classical (high-temperature) conducting material placed in a magnetic field B along the \hat{z} -direction and subject to the voltage V from a battery along the x -direction, as shown in Fig. 1. In 1879, Edwin Hall discovered that under such conditions, a transverse voltage is generated along y . To derive its magnitude, consider that the electrons are subject to an electromotive force plus a Lorentz force (boldface indicating vectors), $F_{tot} = -e\mathbf{E} - e\mathbf{v} \times \mathbf{B}$. The resulting curved trajectories deposit a net buildup of electrons on one side of the sample, which in turn generates an opposing electric field E_y that cancels the Lorentz force. The equilibrium condition is $F_{mag} + F_{Hall} = 0$, or

$$E_{Hall} = v_x B_z. \quad (2.1)$$

With the traditional current density² $j_x = n(-e)v_x$, we have

$$E_{Hall} = \frac{j_x}{-ne} B_z \quad (2.2)$$

¹ h is Planck's constant, e is the magnitude of the charge of an electron

² n is the number density of charge carriers; see any standard text for reference

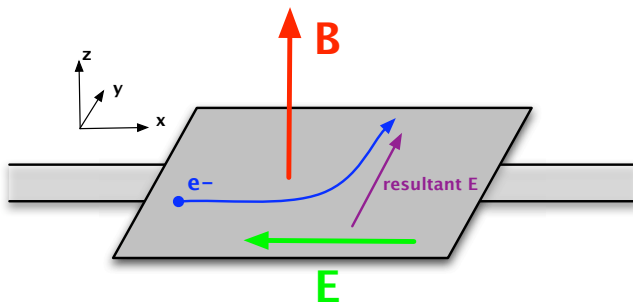


FIGURE 1. The classical Hall effect in a metal

yielding the Hall “resistance”, defined as

$$\begin{aligned} R_{Hall} &= \frac{V_{Hall}}{I_x} = \frac{E_{Hall}}{j_x} \\ &= \frac{-B}{ne}, \end{aligned} \quad (2.3)$$

which, surprisingly, is independent of any physical parameter except the density of charge carriers. Note that R_{Hall} is not a true electrical resistance, because it involves the (transverse) Hall voltage.

In the quantum system of the IQHE, we will need a more robust formalism. I quote the result now, which is a charge carrier density $n = ieB/h$ with i a positive integer. Plugging this into Eq. (2.3) already yields a quantized Hall resistance,

$$R_{Hall} = \frac{h}{ie^2}, \quad i = 1, 2, \dots \quad (2.4)$$

This tells us that the Hall resistance is one of these values; *which* one it is depends on the quantum statistics of the system, as discussed in the next section.

3. LANDAU LEVELS AND THE DENSITY OF STATES

Any independent quantum mechanical system can be fully described by solving the Schroedinger equation using a Hamiltonian that describes all the appropriate physical conditions. If you are unfamiliar with all the mathematical details involved, you may wish to skip ahead to the answer, Eq. (3.11).

3.1. The Hamiltonian. The Hamiltonian for an electron that is completely free in two dimensions, x and y , is given by

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 + e\phi \quad (3.1)$$

where $\mathbf{p} = \mathbf{p}_x + \mathbf{p}_y$, and $-e$ and m are the charge and mass of an electron. Recall that you can choose any scalar potential ϕ and vector potential \mathbf{A} so long as the electric and magnetic fields are generated correctly (recall: $\mathbf{B} = \text{curl } \mathbf{A}$ and $\mathbf{E} = -\text{grad } \phi$). In our case it is convenient to choose

the Landau gauge, $\phi = 0$ and $\mathbf{A} = Bx\hat{y}$, yielding $\mathbf{B} = \nabla \times \mathbf{A} = B\hat{z}$. Now we have

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 = \frac{1}{2m}[(\mathbf{p} \cdot \mathbf{p}) - (e\mathbf{p} \cdot \mathbf{A}) - (e\mathbf{A} \cdot \mathbf{p}) + (e^2\mathbf{A} \cdot \mathbf{A})] \quad (3.2)$$

$$\begin{aligned} &= \frac{1}{2m}(\mathbf{p} \cdot \mathbf{p} - 2e(Bx\hat{y} \cdot \mathbf{p}) + e^2\mathbf{A} \cdot \mathbf{A}) \\ &= \frac{1}{2m}(p_x^2 + p_y^2 - 2eBxp_y + e^2B^2x^2), \end{aligned} \quad (3.3)$$

where I am justified in contracting the cross-terms because the commutator of A and p is zero, which you can verify. Don't be confused about which direction is x and which is y . As long as we're talking only about the quantum mechanics, and not about the Hall effect, there's nothing special about either one — the electron sheet is symmetric around the z -axis. However, we will solve the Schroedinger equation differently for x and y , simply because that's what works out conveniently for our choice of vector potential.

3.2. Solving the Schroedinger Equation. By separating variables, I can write

$$\hat{H}_{x,y}\psi(x, y) = E\psi(x, y)$$

as

$$\hat{H}_x\psi(x)\hat{H}_y\psi(y) = E\psi(x)\psi(y), \quad (3.4)$$

and then solve for the eigenstates of x and y separately. In our Hamiltonian (3.3), the coordinate y does not appear explicitly, and thus we know the momentum p_y is conserved. If the momentum does not change, then we essentially have a free particle in the y direction, and we know its eigenstates are plane wave solutions,

$$\psi(y) = \psi_0 e^{\frac{ip_y y}{\hbar}}, \quad (3.5)$$

with eigenvalues $E = p_y^2/2m$. As is common practice, we can assume the wavefunctions taper smoothly toward the edges of the sample; mathematically, this is realized by imposing periodic boundary conditions in y , so that

$$e^{\frac{ip_y y}{\hbar}} = e^{\frac{2\nu i\pi y}{L_y}}, \quad \nu = 1, 2, \dots \quad (3.6)$$

yielding the allowed values of p_y :

$$\frac{p_y}{\hbar} = \frac{2\nu\pi}{L_y} \quad (3.7)$$

which will become important later. After plugging (3.5) into (3.4), we obtain a Schroedinger equation entirely in x :

$$-\frac{\hbar^2}{2m} \frac{\partial^2 X}{\partial x^2} + \frac{1}{2m} (p_y^2 - 2eBp_y x + e^2 B^2 x^2) X = EX. \quad (3.8)$$

which looks more familiar written this way:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 X}{\partial x^2} + \frac{1}{2} m\omega_c^2 (x - x_0)^2 X = EX \quad (3.9)$$

which is just the equation for a one dimensional harmonic oscillator centered on x_0 . Here

$$x_0 = \frac{p_y}{eB} \quad (3.10)$$

and $\omega_c = \frac{eB}{m}$ is the *cyclotron frequency*, after its role in the behavior of a classical particle in a magnetic field. The harmonic oscillator solutions, which can be found in any standard text, are a discrete spectrum of energy levels

$$E_{i,x} = \hbar\omega_c\left(i + \frac{1}{2}\right), \quad i = 0, 1, 2, \dots \quad (3.11)$$

which, in cases involving a quantum fluid of electrons, are called ‘‘Landau’’ energy levels. (L. D. Landau was the first to solve this problem.) The corresponding wavefunctions are the usual ones involving the Hermite polynomials H_i :

$$\psi_{n,\nu}(x, y) = \psi_n(x)\psi_\nu(y) = e^{\frac{2\nu i\pi y}{L_y}} \sqrt{\frac{1}{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\left(\frac{1}{2}\frac{m\omega x^2}{\hbar}\right)} H_i\left(\sqrt{\frac{m\omega x^2}{\hbar}}\right)$$

which you can promptly forget.

3.3. Capacity of Levels. In this Landau gauge, the solutions for the allowed electron states can be thought of as being extended over the entire sample in the y direction, and as a localized harmonic oscillator eigenstate in x , centered on the point $x_0 = p_y/eB$ (Eq. 3.10). Treating the electron layer as a single system, we can now ask how many electrons are allowed in each Landau level, since by the Pauli exclusion principle there cannot be more than one electron in the same state and the same position in space. Mathematically, this condition can be imposed by demanding that the center of the electron wavefunction lies inside the sample. Recalling (3.10) and (3.7):

$$x_0 = \frac{p_y}{eB} = \frac{2\nu\hbar\pi}{eBL_y}. \quad (3.12)$$

The condition that must be satisfied is

$$0 \leq x_0 \leq L_x,$$

which, when $x_0 = L_x$, places an upper bound on ν ,

$$x_0 = \frac{2\nu_{max}\hbar\pi}{eBL_y} = L_x$$

or

$$\nu_{max} = \frac{L_x L_y eB}{2\pi\hbar} = \frac{L_x L_y}{2\pi l^2}, \quad (3.13)$$

where $l = \sqrt{\frac{\hbar}{eB}}$ is the *magnetic length*, a convenient length scale for this problem. The index ν determines how many y eigenstates (the extended direction) we can fit into each allowed x energy level. Since each y eigenstate only allows one electron (two if we consider spin), this is equivalent to saying how many electrons can be fit into said energy level. We can now ask how many electrons will fit in each level by considering the case when an integer number of levels are filled. Here the total number of electrons in the sample $N = nL_x L_y$ is equal to the number of filled levels i_{filled} times number of electrons allowed per level, ν_{max} . Thus

$$n = \frac{i_{filled}\nu_{max}}{L_x L_y} = i_{filled} \frac{eB}{h}, \quad i = \text{integer Landau index} \quad (3.14)$$

(not $\hbar!$), where ν_{max} is the highest allowed eigenvalue of momentum along y . Thus in a sample of finite size, we can fit exactly eB/h electrons per Landau level. (Technically, there are $2eB/h$ per level, taking into account that electrons have two spin states.) Like the Hall resistance, and just as surprising, this does not depend on any physical parameters of the system!

3.4. Density of States. We now have a spectrum of energy (Landau) states spaced $\hbar\omega_c$ apart, each of which can hold eB/h electrons. Since electrons like to fall into the lowest-energy configuration, our sheet of N electrons will fill up the lowest energy state with eB/h electrons, then the next-lowest energy with another eB/h , and so forth until all the electrons are used up. Fig. 2 shows a scenario where there are exactly enough electrons to completely fill two Landau levels. The energy of the highest electron is the *Fermi energy*, ϵ_F . The top of the distribution will be slightly thermally broadened (not shown), but at low temperatures we can safely assume the spacing between Landau levels is much greater than the mean thermal energy of any electron, i.e. $\hbar\omega_c \gg k_B T$ where k_B is Boltzmann's constant. This is why we are required to cool our system to liquid Helium temperature, about 4 Kelvin, before we begin the experiment.

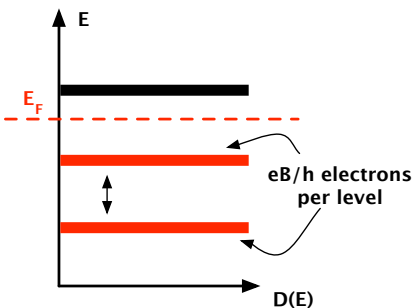


FIGURE 2. Density of states for an ideal two-dimensional system

Suppose now we change the capacity of each Landau level, say, by changing the magnetic field B . As more electrons fit into lower energy levels, the location of the Fermi energy will stay put at the second energy level. At the moment the last electron falls out of this state into the lower band, the Fermi level will suddenly drop to the lower band. Thus, as the magnetic field is altered, the system periodically undergoes abrupt changes in how the electrons are distributed among their allowed modes.

4. OBSERVING THE HALL EFFECT: CONDUCTIVITY AND THE ROLE OF IMPURITIES

In the experiment, we measure the Hall resistance (the voltage along y with respect to a current along x). For most values of B , it appears to vary in a predictable way with the magnetic field, as in the classical case, Eq. (2.3). The peculiar behavior emerges at values that correspond to the exact filling of some number of Landau levels. The first data demonstrating these facts was published by Klaus von Klitzing and colleagues in 1980, and is reproduced in Fig. 3. The discovery won Klitzing the 1985 Nobel prize in Physics. Two questions must be addressed:

- (1) What is the Hall resistance at integer numbers of filled Landau levels?, and
- (2) Why does the resistance jump to the Landau value for nearby values of B , thus creating plateaus?

4.1. Conductivities. To answer the first question, we can reconsider the classical Hall problem in light of a primitive yet surprisingly robust model, introduced by Paul Drude in 1900. The primary assumption of the *Drude model* is that the conducting electrons, which move freely about

the sample, will collide elastically with atoms in their surrounding atomic lattice with probability per unit time $1/\tau$, regardless of their velocity. (In our isolated two-dimensional sheet of electrons, the scattering centers are the impurities in the semiconductor that jut out into the layer.) It is assumed that the electrons, when they do bounce off an impurity, will scatter into a different state. But when an exact number of Landau levels are filled, there are no vacant states available to an electron unless it jumps to a higher Landau level — not likely, considering the low temperature! In this case the relaxation time, τ , goes to infinity.

What happens in a Drude system with a very long relaxation time? Specifically, we would like to know about the electrical conductivity when some number of Landau levels are exactly filled. To do this, we look for a steady-state solution of the Drude equation of motion (which is derived from first principles in many standard texts), using the classical Hall forces:

$$0 = -\frac{\mathbf{p}(t)}{\tau} + (-e)\left(E + \frac{\mathbf{p}}{m} \times \mathbf{B}\right)$$

which we separate into equations in x and y , and rearrange to find the resistivity tensor equation $\mathbf{E} = \|\rho\| \mathbf{j}$:

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \frac{m}{ne^2\tau} \begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix} \begin{pmatrix} j_x \\ j_y \end{pmatrix} \tag{4.1}$$

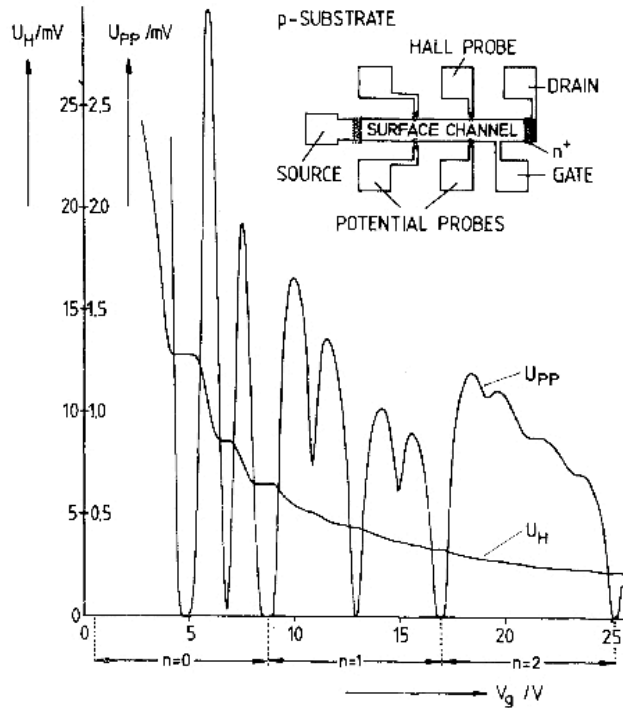


FIGURE 3. Source: Klitzing, Dorda, and Pepper, Phys. Rev. Lett. 45, 494-497 (1980)

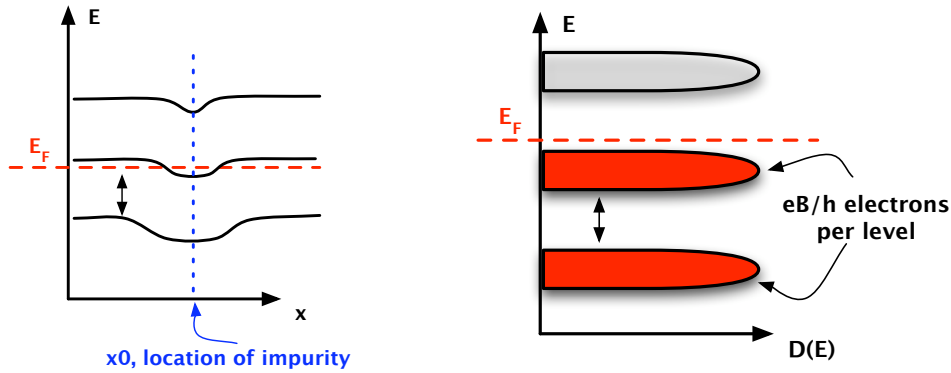
with \mathbf{j} the current density and ω_c the cyclotron frequency, eB/m . When an integer number of Landau levels are exactly filled, $\tau \rightarrow 0$ and the resistivity tensor reduces to:

$$\lim_{\tau \rightarrow \infty} \|\rho\| = \begin{pmatrix} 0 & \frac{B}{ne} \\ -\frac{B}{ne} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{h}{ie^2} \\ -\frac{h}{ie^2} & 0 \end{pmatrix}, \quad i = 1, 2, \dots \quad (4.2)$$

where the last step reflects a quantum system with Landau energy levels (invoking Eq. 3.14). So, when i Landau levels are exactly filled, the Hall conductivity (the off-diagonal elements of $\|\rho\|$) is a constant function of B , and its value is given by ie^2/h . At the same time, the forward “magneto-conductivity” (on-diagonal elements) is zero.

4.2. A Realistic System. As I showed in section 3, in a perfect two-dimensional sheet of electrons, all energy levels are confined to the exact integer Landau energies. In a real-life system, where it is not so easy to manufacture a perfect system, this picture is not quite correct. As I mentioned before, there are often impurities in the surrounding semiconductor material that jut into the electron layer. These impurities introduce local variations in the energy levels, and many can even grab electrons and keep them bound in so-called *localized states*. Those states don’t conduct electricity!

The effect of an impurity is shown in Fig. 4(a), which is to alter the nearby potential energy and thereby change the mechanics of how electrons fill up Landau levels.



(a) Impurities alter the local energy potential

(b) Broadened density of states

FIGURE 4. A realistic system, showing the effect of impurities

Such a fluctuation could be either a decrease (shown) or an increase in the energy of the level, resulting in a broadening of the Landau levels as shown in the new density of states, Fig. 4(b). The essence of this result is that there are an extra number of available electron energy states just above and below each Landau level that do not conduct electricity.

Suppose now one wants to measure the Hall conductivity as the Fermi energy is lowered from fully above one Landau level to fully below it, as shown in Fig. 5. At the beginning, localized electron states with higher energies than the integer Landau energy are being depopulated, resulting in no change to the number of conducting electrons in that level. During this time, the conductivity remains constant. Next, and for most of the transition, electrons are being removed from the extended states, which do conduct electricity, so the conduction of the sample decreases proportionally

with the decreasing Fermi level. Toward the end, all the conduction electrons have disappeared from this Landau level, but localized states are still being depopulated, and the conductivity is again flat, this time at a lower value.

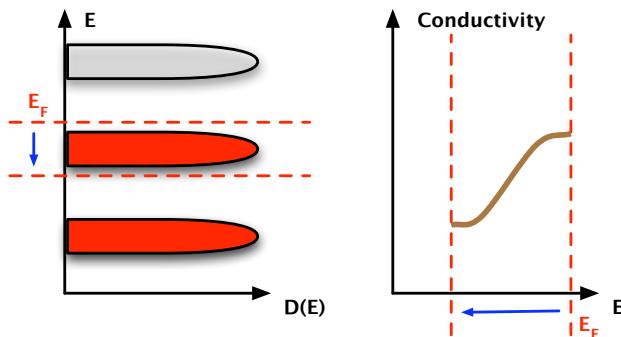


FIGURE 5. Effect of lowering the Fermi energy through one broadened Landau level

In practice, the Fermi energy is effectively lowered by increasing the magnetic field, thereby increasing the capacity of the lower Landau levels. Thus, as we vary B , we expect to see plateaus at the values h/e^2 (actually $h/2ie^2$ if we consider spin degeneracy; see the note after Eq. 3.14).

5. ON THE EXTREME ACCURACY OF THE QUANTUM HALL EFFECT: LAUGHLIN'S GAUGE PRINCIPLE

We should also expect that some nonlinear effects or other unforeseen quantum mechanical couplings would alter the values near these plateaus, giving it an error that would be depend on exactly how an experimenter prepared his or her sample. However, data indicates the Hall plateaus are routinely accurate to the ratio h/e^2 to a precision better than 1 part in 10^7 ! This kind of accuracy is unusually good, and implies that there must be some kind of deeper significance to the underlying physics. When Klitzing first published his data, condensed matter physicists immediately began working to develop a more fundamental theory to explain this accuracy.

One clue to the strangeness of the behavior is that the electrons appear not to interact with their host semiconductor material in any way, nor are their properties affected by different geometries of material. one can remove any interior part of a quantum Hall system — for instance, by drilling a hole — and the system still behaves just as it did before. Indeed, if we visualize the electron wavefunctions classically, they are making circular orbits at the radius of the “size” of the wavefunction. Neighboring orbits have trajectories in opposite directions, and so the current they carry cancels out entirely. The exception is any electron traveling near the edge of the sample, where the mechanics are more complicated but the result is that not all the current is cancelled. These “edge currents” play a significant role in our understanding.

The year after Klitzing's discovery, R. B. Laughlin published an explanation of the IQHE that is elegant in its ability to predict the effect in a way that is totally insensitive to experimental particulars. (Laughlin would go on to win part of the 1998 Nobel for his explanation of the *fractional* quantum Hall effect; another part went to Horst Stormer, now a professor at Columbia and the person who created this lab.) Laughlin's theory essentially demands that the physical, observable

sheet of electrons obey invariance under transformations of the non-observable electromagnetic gauge. Imagine that the sample is folded under itself along the x axis, so that it forms a continuous loop. Let us now assign the coordinates x and y specific orientations so that the direction in which electron wavefunctions that are extended stretches around the loop (x) and the harmonic oscillator eigenstates are spaced across the ribbon (y). The magnetic field B still points through the face of ribbon everywhere (we can keep the same form for the Landau gauge, only now x is a cyclic coordinate). The setup is shown in Fig. 6.

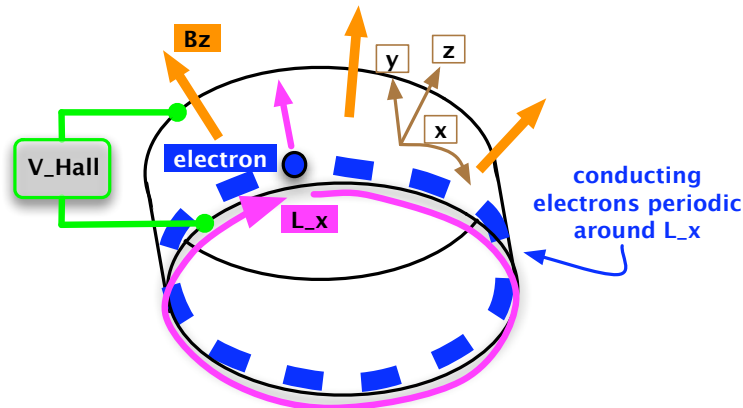


FIGURE 6. A Quantum Hall "Ribbon"

We would like to know what happens when the loop is "threaded" with a unit of vector potential $\Delta\mathbf{A}$:

$$A \rightarrow By\hat{x} + \Delta A\hat{x}.$$

In such case, we have a new Hamiltonian (cf. Eq. 3.2)

$$\begin{aligned} H &= \frac{1}{2m} [\mathbf{p} - e(\mathbf{A} + \Delta\mathbf{A})]^2 - e\phi \\ &= \frac{1}{2m} (p_x^2 + p_y^2 - 2e(By + \Delta A)p_x + e^2(By + \Delta A)^2) \end{aligned} \quad (5.1)$$

yielding the harmonic oscillator part (cf. Eq. 3.9)

$$\frac{-\hbar^2}{2m} \frac{\partial^2 Y}{\partial y^2} + \frac{1}{2} m\omega_c^2 (y - y_0 + \frac{\Delta A}{B})^2 Y = EY$$

and the same relationship between the harmonic oscillator centers and the extended state wavevectors (cf. Eq. 3.10) $y_0 = p_x/eB$. Evidently, such a transformation will cause a shift in the center of each electron's wavefunction in the y direction, across the ribbon, by an amount

$$y_0 \rightarrow y_0 - \Delta A/B \quad (5.2)$$

The threading action also produces a phase shift in the wavefunctions along the extended direction x :

$$\psi(x) = e^{\frac{ip_x x}{\hbar}} = e^{\frac{iy_0 e B x}{\hbar}} \rightarrow e^{\frac{iy_0 e B x}{\hbar}} e^{\frac{-i\Delta A e x}{\hbar}} \quad (5.3)$$

This last result implies that in the direction around the loop, electrons may change their phase, and thereby change the location where they are most likely to be found. Yet the only action we performed was a transformation of the gauge, which is not supposed to have real, observable effects! The only way around this problem is to require that the electron remains unchanged — any shift in vector potential must produce a complete translation around the loop, $x \rightarrow x + L_x$, so that the electron looks the same as it did before:

$$1 = e^{\frac{-i\Delta A e L_x}{\hbar}} \rightarrow \frac{\Delta A e L_x}{\hbar} = 2\pi \quad (5.4)$$

This results in a quantization of the vector potential

$$\Delta A = \frac{\hbar 2\pi}{e L_x} = \frac{h}{e L_x} \quad (5.5)$$

or in terms of the wavefunction centers (5.2):

$$\frac{\Delta A}{B} = \frac{h}{e B L_x} \quad (5.6)$$

But if eB/h defines the number of conducting electrons (Eq. 3.14), then we can show the above equation defines the quantum of vector potential as exactly the spacing between conducting electrons along y ! Thus, for every quanta of magnetic flux added to the system, one electron per Landau level shifts over by one in the y direction. If, however, an electron is bound to an impurity in a localized state, we can show that its wavefunction also picks up a phase shift à la Eq. 5.3, but does not shift location. Hence, the IQHE is an exactly quantized charge pump, and is not disrupted by the presence of localized states, nor is it affected by a hole drilled in the ribbon. In this way, we can say the IQHE is of a topological nature.