Johnson Noise

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An experiment was performed that determines the Boltzmann constant k and the centigrade temperature of absolute zero by measuring the thermal noise of resistors. The Nyquist theorem provides a quantitative relationship between the thermal electromotive force across a conductor and its resistance and temperature. Measurement of the root-mean-square RMS voltage for a variety of resistors at a fixed temperature was used to calculate the Boltzmann constant. The RMS voltage for a 22.5 kΩ resistor was measured over 300 degree temperature range. This latter data extrapolated to zero centigrade gave an estimate of absolute zero and provided an additional method for determining the Boltzmann constant. The experimentally determined values of the Boltzmann constants, $1.37 \pm 0.06 \times 10^{-23}$ J/K & $1.363 \pm 0.025 \times 10^{-23}$ J/K, and the centigrade temperature of absolute zero, -265.5 ± 6.9°C, are in good agreement with the accepted values.

I. INTRODUCTION

This paper is a full report on the junior lab experiment: **Johnson Noise**. In this experiment, we study the phenomenon of thermal (Johnson) noise as predicted by the Nyquist Theory.

This report has been partitioned into sections accordingly, each discussing a specific aspect of the experiment. Section II discusses the theoretical background relevant to the experiment by deriving the Nyquist Theorem using two different approaches. The experimental apparatus and details of its operation are discussed in section III. Section IV presents the experimental results. Concluding remarks are given in section V.

II. NYQUIST THEORY

Johnson Noise is the mean-square electromotive force in conductors due to thermal agitation of the electromagnetic modes which are coupled to the thermal environment by the charge carriers. The Nyquist Theory is of great importance to experimental physics and in electronics. It gives a quantitative expression for the Johnson Noise generated by a system in thermal equilibrium and is therefore needed in any estimate of the limiting signal-to-noise ratio of an experimental apparatus. In this section, the Nyquist theorem is derived in two ways: first, following the original transmission line derivation, and, second using microscopic arguments [1], [2].

A. Transmission Line Derivation

Consider two conductors each of resistance R at a temperature T connected as depicted in Figure 1. Conductor 1 produces a current I in the circuit equal to the electromotive force due to thermal agitation divided by the total resistance $2R$. This current delivers power to conductor 2 equal to current squared times the resistance. By symmetry, one can deduce that the situation is reciprocal. Conductor 2 produces a similar current which delivers power to conductor 1. Because the two conductors are at the same temperature, the second law of thermodynamics dictates that the power flowing in both directions is equal. I emphasize that no assumption about the nature of conductors has been made.

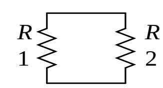

FIG. 1. Two conductors with equal resistance R.

It can be shown that this equilibrium condition holds at any given frequency. Suppose there exists a frequency interval $\Delta\nu_1$ where conductor 1 receives more power than it transmits. We then connect a non-dissipative network with a resonance in the frequency interval $\Delta\nu_1$ between the two conductors (refer to Figure 2). Since the system was in equilibrium prior to inserting the network, it follows that after is insertion more power would be transferred from conductor 2 to conductor 1. However, as the conductors are at the same temperature, this would violate the second law of thermodynamics. The results we have arrived at are important enough to merit summarizing. By eminently reasonable theoretical arguments, we can conclude that the electromotive force due to thermal agitation in conductors are *universal* functions of (refer to Figure 3):

- frequency ν
- resistance R
- temperature T

Experiments performed by Dr. J. B. Johnson in 1928 confirmed the formula which was later derived Dr. H. Nyquist on purely theoretical grounds [3].

The derivation of the mean-square voltage $\langle V^2 \rangle$ across a conductor closely follows Nyquist's original derivation. The problem of determining a quantitative expression for the thermal agitation (i.e. the mean-square voltage) of

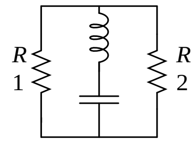

FIG. 2. Two conductors plus resonant circuit.

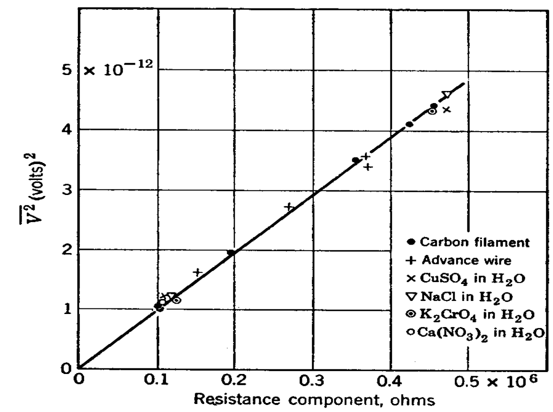

FIG. 3. Voltage-squared vs. resistance component for various types of conductors.

the conductor can be viewed as a simple one-dimensional case of black-body radiation. Consider a lossless one-dimensional transmission line of length L terminated at both ends by conductors with resistance R. The transmission line has been chosen to have a characteristic impedance $Z = R$; consequently any voltage wave propagating along the transmission line is completely absorbed by the terminating resistor without any reflections. Voltage waves of the form $V = V_0 \exp\left[i(k_x x - \omega t)\right]$ propagate down the transmission line at velocity $v = \omega/k_x$. The available number of modes can be calculated by imposing the periodic boundary condition $V(0) = V(L)$ on the propagating voltage waves. The wave vector k_x is related to the length by the relation $k_x L = 2\pi n$ where n is any integer. The density of modes is then,

$$\begin{aligned} D(\omega) &= \frac{1}{L}\frac{dn}{d\omega} \\ &= \frac{1}{L}\frac{dn}{dk_x}\frac{dk_x}{d\omega} \\ &= \frac{1}{2\pi v} \end{aligned} \tag{1}$$

The mean energy per mode is given by the Planck formula,

$$\langle \varepsilon(\omega)\rangle = \frac{\hbar\omega}{\exp\frac{\hbar\omega}{kT} - 1} \tag{2}$$

$$\langle \varepsilon(\omega)\rangle \approx kT \tag{3}$$

where in the last line we made use of the equipartition theorem: in the classical limit, $\hbar\omega \ll kT$, each squared canonical term in the the Hamiltonian contributes $\frac{1}{2}kT$ to the mean energy.[1]

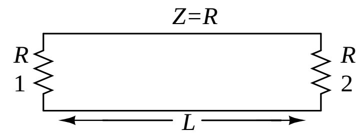

FIG. 4. Lossless transmission line $Z = R$ of length L with matched terminations.

The energy density per unit frequency $U(\omega)$ is then given by the product of the density of modes and the mean energy per mode:[2]

$$\begin{aligned} U(\omega) &= D(\omega)\langle\varepsilon(\omega)\rangle \\ &= \frac{kT}{2\pi v} \end{aligned} \tag{4}$$

The power per unit frequency is then simply:[3]

$$\begin{aligned} P(\omega) &= vU(\omega) \\ &= \frac{kT}{2\pi} \end{aligned} \tag{5}$$

OR

$$P(\nu) = kT \tag{6}$$

This is the power per unit frequency *absorbed* by the resistor. By the *principle of detailed balance* this must be equal to the power per unit frequency *emitted* by the resistor. The thermal electromotive force generated by the resistor sets up a current $I = V/2R$ in the transmission line. Thus, the power absorbed by the resistor at the other end is

$$P(\nu) = \langle I^2(\nu)\rangle R \tag{7a}$$

$$= \left\langle \frac{V^2(\nu)}{4R^2} \right\rangle R \tag{7b}$$

$$= \frac{\langle V^2(\nu)\rangle}{4R} \tag{7c}$$

Equating Eq. 6 to Eq. 7c and then solving for the mean-square voltage per unit frequency gives:

$$\langle V^2(\nu)\rangle = 4RkT \tag{8}$$

By integrating the expression above over the accesible frequency range, we arrive at the Nyquist Theorem:

$$\boxed{\langle V^2\rangle = 4kTR\Delta\nu \qquad (9)}$$

[1]The Hamiltonian (per unit volume) for an electromagnetic wave is given by $\mathcal{H} = \frac{1}{8\pi}(\mathbf{E^2} + \mathbf{B^2})$.

[2]$U(\omega)$ is a one-dimensional energy density.

[3]Recall that the energy density is equivalent to a force.

B. Microscopic Derivation

Consider a conductor of resistance R with a charge carrier density N having a relaxation time τ_c. The conductor has length ℓ and cross-sectional area A. The voltage V across the conductor is

$$
\begin{aligned}
V &= IR && (10a)\\
&= RAj && (10b)\\
&= RANe\langle u\rangle && (10c)
\end{aligned}
$$

where I is the current, j is the current density, e is the charge on an electron, and $\langle u\rangle$ is the drift speed along the conductor.

Noting that $NA\ell$ is the total number of electrons in the conductor,

$$\sum_i u_i = NA\ell\langle u\rangle \tag{11}$$

Solving for $\langle u\rangle$ in Eq. 11 and substituting the resulting expression into Eq. 10c gives,

$$V = \sum_i V_i = \frac{Re}{\ell}\sum_i u_i \tag{12}$$

where u_i and V_i are random variables.

The spectral density $J(\nu)$ has the property that in the frequency interval $\Delta\nu$

$$\langle V_i^2\rangle = J(\nu)\Delta\nu \tag{13}$$

The correlation function can be written as

$$
\begin{aligned}
C(\tau) &= \langle V_i(t)V_i(t+\tau)\rangle && (14a)\\
&= \langle V_i^2(t)\rangle \exp\left(-\tau/\tau_c\right) && (14b)
\end{aligned}
$$

where τ is an arbitrary time interval.

By substiting Eq. 14b and Eq. 12 into the Wiener-Khintchine theorem Eq. 15a, the spectral density is

$$
\begin{aligned}
J(\nu) &= 4\int_0^\infty C(\tau)\cos\left(2\pi\nu\tau\right)d\tau && (15a)\\
&= 4\left(\frac{Re}{\ell}\right)^2\langle u^2\rangle\int_0^\infty exp(-\tau/\tau_c)\cos\left(2\pi\nu\tau\right)d\tau && (15b)\\
&= 4\left(\frac{Re}{\ell}\right)^2\langle u^2\rangle\frac{\tau_c}{1+(2\pi\nu\tau_c)^2} && (15c)\\
&\approx 4\left(\frac{Re}{\ell}\right)^2\langle u^2\rangle\tau_c && (15d)\\
&\approx 4\left(\frac{Re}{\ell}\right)^2\left(\frac{kT}{m}\right)\tau_c && (15e)
\end{aligned}
$$

where $\langle u^2\rangle = kT/m$ by the equipartition theorem. Note that for metals at room temperature $\tau_c < 10^{-13}$, thus from the DC through the microwave range $2\pi\nu\tau_c \ll 1$.

Thus the mean-square voltage in the frequency range $\Delta\nu$ equals:

$$
\begin{aligned}
\langle V^2\rangle &= NA\ell\langle V_i^2\rangle && (16a)\\
&= NA\ell J(\nu)\Delta\nu \quad \text{using Eq. 13} && (16b)\\
&= NA\ell 4\left(\frac{Re}{\ell}\right)^2\left(\frac{kT}{m}\right)\tau_c\Delta\nu \quad \text{using Eq. 15e} && (16c)\\
&= 4\left(\frac{Ne^2\tau_c}{m}\right)\frac{A}{\ell}R^2kT\Delta\nu && (16d)
\end{aligned}
$$

Using a result from conductivity theory $\sigma = Ne^2\tau_c/m$ and the elementary relation $R = \frac{\ell}{\sigma A}$ [5]:

$$\langle V^2\rangle = 4\,\underbrace{\sigma\frac{A}{\ell}}_{1/R}\,R^2kT\Delta\nu \tag{17}$$

We have once again arrived at the Nyquist Theorem:

$$\boxed{\langle V^2\rangle = 4kTR\Delta\nu \qquad (18)}$$

The Nyquist Theorem is a special case of the general connection existing between fluctuations (random variables) and dissipation in physical systems. Brownian motion lends itself to a similar analysis [6], [7].

III. EXPERIMENTS

This section describes the experimental apparatus used, the calibration performed and the measurements that were recorded.

A. Apparatus

Figure 5 is a diagram of the experimental apparatus used to measure the Johnson Noise.[4] An inverted beaker shielded the resistor R which was mounted on the terminal of the aluminum box. The resistor is connected to the measurement chain through two switches (SW1 and SW2). A Hewlett-Packard HP54601A digital oscilloscope was used to measure the root-mean-square (RMS) voltage generated by the resistor. Because the Johnson Noise signals are in the microvolt range, a low-noise amplifier (PAR 113) was used to produce millivolt signals detectable by the digital oscilloscope. A band-pass filter (Krohn-Hite 3202R) was used to prevent thermal noise outside the frequency range 1 KHz – 50 KHz from being amplified.[5]A Tektronix Function Generator (FG) 504

[4]Figure 5 was scanned-in from the junior lab guide [8].

[5]Signals outside this frequency range could not be properly amplified by the PAR 113.

provided sinusoidal calibration signals. The FG and the Kay attenuator were used to calibrate the measurement chain.

Several steps were taken to filter out extraneous noise from the experimental apparatus. At all times the digital oscilloscope was kept at least five feet from the noise source, otherwise the variable magnetic field from its beam-control coil would produce undesirable electrical oscillations in our noise measurements. Coaxial cables were also kept as short as possible to keep minimize electrical interference.

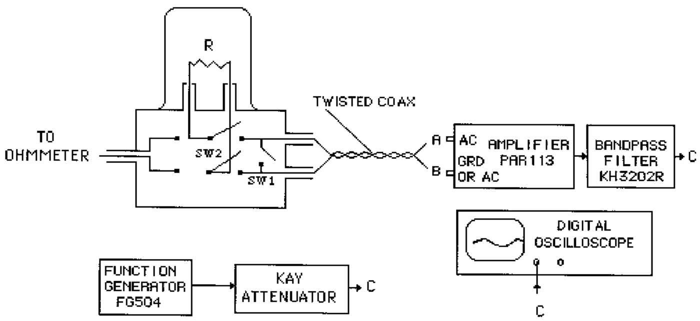

FIG. 5. Experimental apparatus.

B. Calibration of Measurement Chain

1. Test signal RMS voltage

The amplitude of the sinusoidal signal produced from the FG was adjusted so that the RMS voltage V_{RMS} as measured on the digital oscilloscope was approximately 2 volts. The RMS voltage of the FG sinusoidal signal was recorded over the range passed by the Krohn-Hite Filter (refer to Figure 6). It was confirmed that the RMS voltage varied slightly over the frequency range of interest.

2. Gain of measurement chain

The sinusoidal test signal was fed through the Kay attenuator set to 60 dB (1000) of attenuation to the 'A' input of the PAR amplifier (set to 1K) with 'B' input grounded. The RMS voltages out of the Krohn-Hite filter were measured over a 100 kHz frequency range. The gain squared $[g(\nu)]^2$ was small at very low frequency, then drastically increased to unity around 5 kHz (refer to Figure 7). As expected at higher frequency (> 50 kHz) the gain squared roll off considerably.

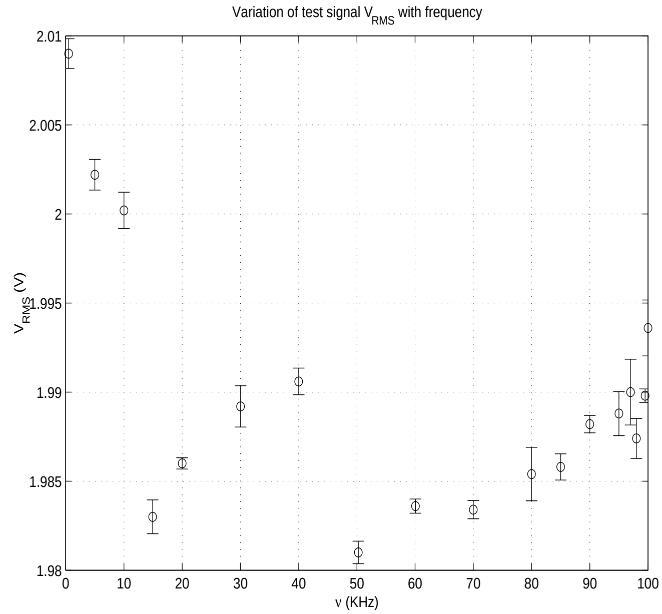

FIG. 6. RMS voltage V_{RMS} produced by function generator as a function of frequency.

C. Resistance Dependence of Johnson Noise

With the PAR amplifier set to 1K, typical RMS voltages out of the Krohn-Hite filter were in the millivolt range. The component of the noise V_S not generated by the resistor but by the amplifier itself was measured by:

1. Opening SW2.
2. Unplugging the connections to the ohmmeter and temperature meter.
3. Shorting the resistor with SW1.

The total RMS voltage V_R was measured with the shorting switch SW1 open. Because all the contributions to the measure RMS voltage are statistically uncorrelated, *they add in quadrature*. Thus, mean square Johnson noise of the resistor is given by,

$$V_{\mathrm{Jo}}'^2 = V_R^2 - V_S^2 \tag{19}$$

where V_R and V_S are the RMS voltages measured with the SW1 open and closed, respectively. The resistance R was measured using a digital multimeter after each noise measurement.

D. Temperature Dependence of Johnson Noise

The Johnson noise of a 22.2 kΩ resistor was measured at liquid N_2 temperature −160°C to 150°C. High temperatures were obtained by mounting the inverted aluminum box and placing it on a cylindrical oven. The temperature was adjusted by using a Variac. Low temperatures

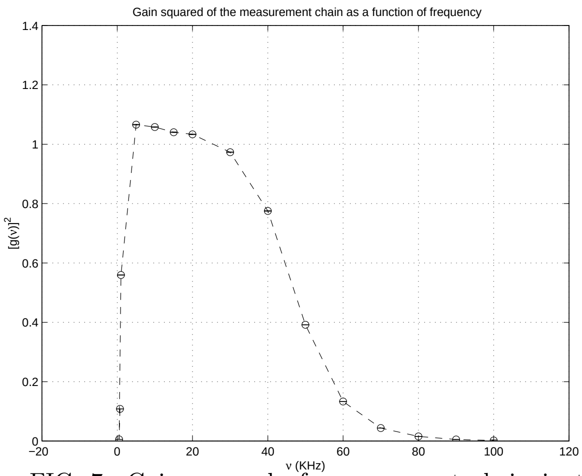

FIG. 7. Gain squared of measurement chain in the frequency range (0.5 KHz – 100 kHz.) **NOTE:** The dotted line is not a fitted function. Its purpose is tom emphasize a trend in the gain squared. The gain squared has been normalized such that the value of $[g(\nu)]^2 = 1$ corresponds to a gain of 1000.

were obtained by inverting the aluminum box and placing it on a liquid N_2 filled dewar flask. The temperature was varied in an *ad-hoc* manner by raising and lowering the aluminum box into the dewar flask as needed.

IV. RESULTS AND DISCUSSION

The first subsection explicitly connects the Nyquist Theorem with the experimental setup at hand. The last two subsections describe the results of the subsections III C & III D, respectively.[6]

A. Derivation of RMS thermal voltage at the terminal of an RC circuit

The resistor and coaxial cables that are connected to the PAR amplifier can be modeled as the circuit depicted in Figure 8. The equivalent circuit is composed of a fluctuating thermal electromotive force V_{Jo} with an ideal resistor R and a capacitor C in a simple lowpass filter configuration.

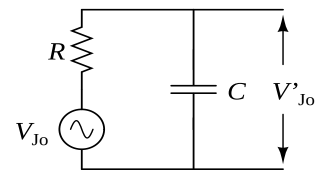

FIG. 8. Equivalent circuit of the electromotive force across a conductor of resistance R connected to the measuring device with cables having capacitance C.

In sinusoidal steady state, impedances can be used to treat the circuit as a voltage divider.

$$V'_{\mathrm{Jo}} = \frac{(i\omega C)^{-1}}{(i\omega C)^{-1} + R} g(\omega) V_{\mathrm{Jo}} \tag{20a}$$

$$= \frac{1}{1 + i\omega C} g(\omega) V_{\mathrm{Jo}} \tag{20b}$$

The RMS thermal voltage is the magnitude of Eq. 20b:

$$V'^2_{\mathrm{Jo}} = \frac{[g(\nu)]^2 V^2_{\mathrm{Jo}}}{1 + (2\pi\nu RC)^2} \tag{21}$$

The Johnson Noise is equation Eq. 21 summed over the accessible frequencies,

$$V'^2_{\mathrm{Jo}} = V^2_{\mathrm{Jo}} \underbrace{\int_0^\infty \frac{[g(\nu)]^2}{1 + (2\pi\nu RC)^2}\, d\nu}_{G} \tag{22}$$

In this experiment, the integral in Eq. 22 was numerical evaluated using the data collected in the calibration of the measurement chain (Figure 7). The capacitance C was approximated at 60 pF from considerations of the amount of coaxial cable used and its known capacitance per unit length, 30.8 pF/feet. The Nyquist Theorem expressed in terms of the present variables is arrived at by taking Eq. 9(or 18) and making the substitutions: $\langle V^2 \rangle \to V'^2_{\mathrm{Jo}}$ and $\Delta\nu \to G$.

$$V'^2_{\mathrm{Jo}} = 4kTRG \tag{23}$$

B. Determination of the Boltzmann Constant

The RMS voltage was measured for eight metal film resistors (whose values ranged from 20 kΩ to 10^3 kΩ) at room temperature. Figure 9 is a plot of V'^2_{Jo} against R. The Boltzmann constant was calculated by solving for k in Eq. 23. The experimentally determined value of the Boltzmann constant, $1.37 \pm 0.06 \times 10^{-23}$ J/K, is in good agreement with the accepted value 1.38×10^{-23} J/K.

[6]Note that Boltzmann constant is calculated in the last *two* subsections.

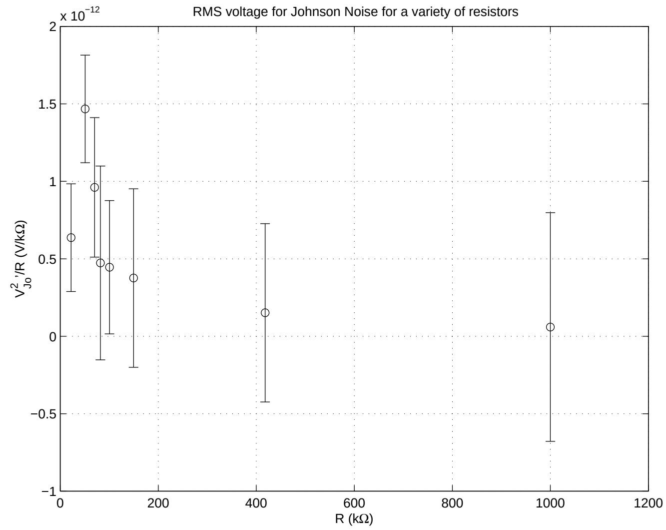

FIG. 9. Resistance dependence of Johnson Noise V'_{Jo}.

C. Determination of the Absolute Zero on Centigrade Scale

The RMS voltage for 22.2 kΩ resistor was measured at fourteen temperatures ranging from $\sim -160°$C to $\sim 150°$C at approximate intervals of 25°C Figure 10 is a least-squares fit of $V_{\mathrm{Jo}}'^2/4RG$ vs. T. The slope of the line gives the Boltzmann constant and the T-intercept is the centigrade temperature of absolute zero. The Boltzmann constant was determined to be $1.363 \pm 0.025 \times 10^{-23}$ J/K and centigrade temperature of absolute zero was extrapolated to $-265.5 \pm 6.9°$C. Both experimentally determined values are in good agreement with their accepted values of 1.38×10^{-23} J/K and -273.15K, respectively.

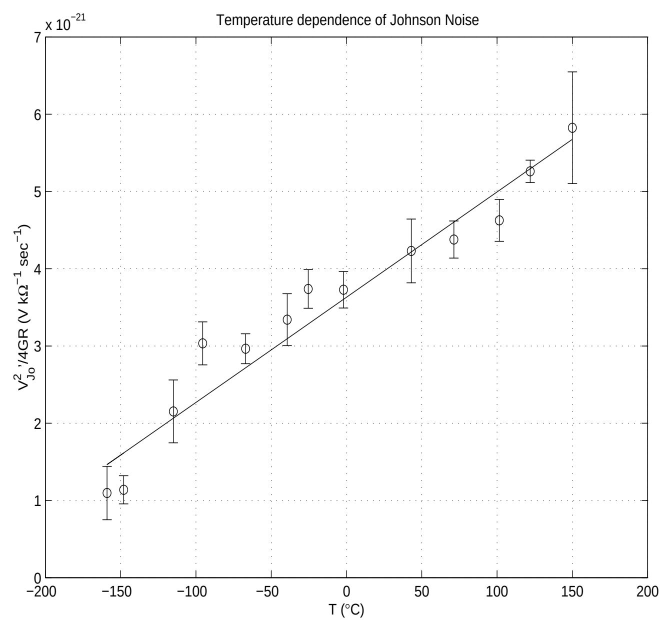

FIG. 10. Temperature dependence of Johnson Noise V'_{Jo}.

V. CONCLUSIONS

Johnson Noise belongs to a broader category of stochastic phenomena which have been of research interest for decades. Measurement of the thermal noise in resistors provided a means to calculate the Boltzmann constant and the centigrade temperature of absolute zero. Because there are inherent difficulties in measuring thermal noise, the Boltzmann constant was measured to an accuracy of ~ 4 %.[7] Alternate methods of implementing a undergraduate physics experiment on Johnson Noise are described in the literature (e.g. [9]).

ACKNOWLEDGMENTS

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