Physics G6038: Prob Set 3 Feb. 20

3.1 The reduced rotation matrix elements are $d^j_{m'm}(\beta) = \langle jm'|e^{-i\beta J_y}|jm\rangle$ and given by the Wigner formula derived in class (p.223 Sakurai).

a) Using only the matrix elements of $J^{\pm}$ compute the 2×2 matrix $d^{\frac{1}{2}}(\beta)$ and compare to $e^{-i\beta\sigma_y/2}$ where σ_y is the Pauli spin y matrix. Show that the Wigner formula gives the same elements.

b) Compute as above the 3×3 reduced rotation matrix in the $j = 1$ irreducible representation and compare to the Wigner.

c) A beam of neutral spin 1/2 and 1 particles with the same g factor ($\vec{\mu} = g\mu_B\vec{s}$) and equal intensity enters a Stern-Gerlach device along the $\hat{y}$ axis with magnetic field gradient in the $\hat{z}$ direction. The initial spin state of both types of particles is prepared to be in the maximal eigenstate of $\hat{n}\cdot\vec{S}$ along direction $\hat{n} = (\theta, \phi)$. Compute the relative intensity of the five beams emerging from the apparatus as a function of θ, ϕ.

d) Derive the relative intensity of the emerging beams for a spin 3/2 beam incoming with "helicity" $-\frac{1}{2}$, i.e. with $S_y|\psi\rangle = -\frac{1}{2}|\psi\rangle$. Use the Wigner formula to compute the rotation matrix elements.

e) Show that for any j, the $(2j+1)\times(2j+1)$ matrix $R_z(\phi) = \exp(-i\phi J_z)$ can be expressed in terms of a finite sum $R_z(\phi) = \sum_{n=0}^{2j} c_n J_z^n$. (Hint: Vandemonde determinant may be useful.) Now apply a $\pi/2$ rotation about the $-\hat{x}$ axis to show that $R_x(-\pi/2)R_z(\phi)R_x(\pi/2) = R_y(\phi) = \sum_{n=0}^{2j} c_n J_y^n$ in terms of the same coefficients c_n. From matrix elements of J_y^n use this method to calculate the $d^j_{m',m}(\beta)$ matrix elements for $j = 1/2, 1$ and compare to parts a,b.

3.2 a) Show that $Y_{lm}(\theta,\phi) = c_l D^l_{0m}(-\psi,-\beta,-\phi) = c_l e^{im\phi} d^l_{m0}(\beta)$, where $c_l = ((2l+1)/4\pi)^{\frac{1}{2}}$.

b) Show that $Y_{l_1m_1}(\theta,\phi)Y_{l_2m_2}(\theta,\phi) = \sum_{lm} c(l_1m_1l_2m_2lm)Y_{lm}(\theta,\phi)$ and determine the coefficients in terms of Clebsch-Gordan ones.

c) Using tables of C-G coefs or direct computation for extra credit, expand $(Y_{10}(\Omega))^2$ in terms of $Y_{lm}(\Omega)$. Repeat for the angular function $f(\Omega) = (x^2 - y^2)/(x^2+y^2+z^2)$.

d) Show that the operator $x^2 = r^2\sum_{lm} c(lm)Y_{lm}(\Omega)$ and determine the coefs. What does this imply about the matrix elements between states $|lm\rangle$ and $|l'm'\rangle$?

3.3 a) Compute all the Clebsh-Gordan coefficients relevant for coupling $j_1 = l$ and $j_2 = 1/2$. (Hint: one way is to use the projection operator method $P(j_0) = \Pi_{j\neq j_0}(J^2 - j(j+1))/(j_0(j_0+1) - j(j+1)))$

b) Evaluate $\vec{L}\cdot\vec{S}|l,\frac{1}{2}, j = l\pm\frac{1}{2}, m\rangle$ where $\vec{J} = \vec{L}+\vec{S}$ and the above state is an eigenstate of $S^2 = 3/4$, $L^2 = l(l+1)$, J^2 and J_z.