Physics G6038: Prob Set 4 Feb. 29

4.1 In problem 2.2e you showed that the character of the $2j+1$ irreducible representation $D^j(\vec{\phi})$ of the rotation group was $\chi^j(\phi) = \mathrm{Tr}D^j(\vec{\phi}) = \sum_{m=-j}^{j} e^{-im\phi} = (e^{-ij\phi} - e^{i(j+1)\phi})/(1-e^{i\phi})$ using the invariance of the trace to similarity transformations and the diagonal form of $e^{-i\phi J_z}$ in the $|jm\rangle$ basis. Consider now the direct product representation $D^{j_1\otimes j_2} = D^{j_1} \otimes D^{j_2}$ with $j_1 \geq j_2$.

a) Compute the character of the direct product representation.

b) Derive the Clebsh-Gordan decomposition

$$\chi^{j_1\otimes j_2} = \chi^{j_1+j_2} + \cdots + \chi^{j_1-j_2}$$

using only the characters of the known irreducible reps.

4.2 Three quarks with spin 1/2 and flavors u,d,s are coupled to form states $|J,m,S\rangle_c$, where $\vec{J} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$ is the total spin and $\vec{S} = \vec{S}_1 + \vec{S}_2$ is the spin of the ud diquark system.

a) What is the range of allowed J and S?

b) Construct the normalized state $|3/2, 1/2, 1\rangle_c$ in terms of the uncoupled basis $|m_u m_d m_s\rangle$.

c) Construct the normalized $|1/2, 1/2, 1\rangle_c$ and $|1/2, 1/2, 0\rangle_c$ states. (Hint: apply projection operators for fixed J^2 and S^2 to suitable states)

4.3 For a proton of spin $\frac{1}{2}$, the s quark above is replaced by a u quark. Because the color wavefunction is assumed to be totally antisymmetric (recall why we assume this because of the $\Delta^{++}(1236)$ resonance), the total space, flavor and spin wavefunction of the quarks must be symmetric under the interchange of any two quarks.

a) Expand the proton with spin projection $+1/2$ wavefunction as a linear combination of states such as $|u\uparrow, u\downarrow, d\uparrow\rangle$.

b) Assuming $m_u = m_d = m_p/3$ is the mass of the constituent quarks and their charge is $e_u = 2e/3, e_d = -e/3$, compute in (ev/gauss) the magnetic moment the proton $\mu = \sum_i g_i e_i/(2m_i c) S_{zi}$. Assume $g_i = 2$. Compare to the observed magnetic moment.

4.4 An irreducible tensor operator T^k_q of order k transforms under rotations as $U(R)T^k_q U(R^{-1}) = \sum_{q'=-k}^{k} T^k_{q'} D^k_{q'q}(R)$ where $U(R) = e^{-i\vec{\phi}\cdot\vec{J}}$ and $D^k(R)$ is a $2k+1$ dimensional irreducible representation of the rotation group.

a) Derive from above the required commutation relations between T^k_q and the generators $J_\pm, J_z$.

b) If $\mathbf{A}$ and $\mathbf{B}$ are two vector operators, prove that the scalar product $\mathbf{A}\cdot\mathbf{B} = -A^1_1 B^1_{-1} - A^1_{-1} B^1_1 + A^1_0 B^1_0$ is rotation invariant.

c) Construct the five components of the quadrapole tensor T^2_q constructed out of vector operator components V_x, V_y, V_z, (e.g. $T^2_2 \propto (V_x + iV_y)^2$)

d) Show that the nine component Cartesian (direct product) tensor $T_{ij} = V_i U_j$ can be decomposed into a sum of irreducible Cartesian tensors

$$T_{ij} = \frac{1}{3}(\vec{V}\cdot\vec{U})\delta_{ij} + \frac{1}{2}\epsilon_{ijk}(\vec{V}\times\vec{U})_k + (\tfrac{1}{2}(V_iU_j + V_jU_i) - \frac{1}{3}(\vec{V}\cdot\vec{U})\delta_{ij})$$

where the first term transforms as a scalar and the second as (axial) vector. Show that the five components of the third term transform among themselves under rotations. Construct irreducible spherical tensors via $T^1_m = \sum_{q,p} U^1_q V^1_p \langle 11qp|11jm\rangle$, to show that

$$\begin{aligned}
T^0_0 &= -(\vec{U}\cdot\vec{V})/\sqrt{3} \\
T^1_{\pm 1} &= \frac{i}{\sqrt{2}}(\mp 1)((\vec{U}\times\vec{V})_x \pm i(\vec{U}\times\vec{V})_y)/\sqrt{2}\ ,\quad T^1_0 = \frac{i}{\sqrt{2}}(\vec{U}\times\vec{V})_z \\
T^2_0 &= (3U_zV_z - \vec{U}\cdot\vec{V})/\sqrt{6} \qquad (1)
\end{aligned}$$

4.5 The nuclear potential between a proton and neutron has a tensor component due to pion exchange: $V_T = f(r)S_{12}$, where

$$S_{12} = 2(3\frac{(\vec{S}\cdot\vec{r})^2}{r^2} - S^2)$$

and $\vec{S} = \vec{s}_p + \vec{s}_n$ is the total spin and $\vec{r}$ is the relative coordinate. Show that $S_{12} = (24\pi/5)^{\frac{1}{2}}\mathbf{S}^{(2)}\cdot\mathbf{Y}^{(2)}$ in terms of the irreducible quadrapole tensors $\mathbf{S}^{(2)}_m$ formed out of the S_i and $\mathbf{Y}^{(2)}_m \equiv Y_{2m}(\hat{r})$.