Measurement of Fundamental Constants using Johnson and Shot Noise

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In this paper, we examine the relationship between noise in an electric circuit and three fundamental constants of nature. Using the Nyquist formula, we find a relation between the Johnson noise and the resistance in the circuit. This leads to an estimate of Boltzmann's constant k_B. The measurement of Johnson noise as a function of temperature gives another estimate of k_B, and also a measurement of the Celsius temperature of absolute zero. Finally, the measurement of shot noise from a photodiode gives us a value for the charge of an electron e. Our results all gave the correct order of magnitude for these constants, however, we were often several standard deviations off of the accepted values. This lead us to conclude that there was some amount of systematic error in our system that was unaccounted for in our measurements, and we discuss possible sources for this error.

1. INTRODUCTION

Thermodynamics serves as a link connecting microscopic properties of physical objects and the macroscopic behavior of an ensemble of these objects. When the predictions of thermodynamics are applied to the physics of conductors and circuits, we find a series of relationships between the voltages observed in the circuit and several fundamental constants. These relationships arise due to two types of noise caused by thermal fluctuations in the circuitry. The first type, Johnson noise, is present in any circuit containing a finite resistance. The second type, Shot noise, arises from discrete passages of charge carriers, such as during the emissions of a photoelectron. In the following experiments, we will describe techniques for measuring these two types of noise, and discuss how they relate to three fundamental constants: Boltzmann's constant k_B, the Celsius temperature of absolute zero, and the charge of the electron e.

2. JOHNSON NOISE

2.1. Theory

J. B. Johnson performed the first measurement of the noise that arises from thermal fluctuations within a resistor [1], and using the values obtained in these measurements, Harry Nyquist provided a theoretical description of Johnson noise. The formula obtained in Nyquist's paper [2] for the mean squared voltage arising due to thermally induced currents in the frequency range f_j and $f_j + df$ is

$$dV_j^2 = 4Rk_BTdf \tag{1}$$

Nyquist arrived at this result as follows. The second law of thermodynamics states that there can be no net transfer of heat between two systems in thermal equilibrium. Using this result, we can conclude that the power transferred between two connected resistors due to thermally induced currents must net to zero at any frequency, regardless of the microscopic properties of the resistor. This leads to the results that the voltage due to thermal excitations can depend only on the system's resistance, temperature, and frequency of the current.

Next, we note that the energy of each mode of vibration has energy contained in the electromagnetic field in, for example, the transmission line connecting the two resistors. The Hamiltonian for the system will then contain a term depending on the magnitude of the electric and magnetic fields: $(E^2 + B^2)/8\pi$. We note that since $hf \ll k_BT$ for our experiment, we need not consider a quantum mechanical description of the thermodynamics of the system. The equipartition theorem of thermodynamics states that each term of the Hamiltonian that depends only on the square of a canonical position or momentum variable (i.e. E and B), contributes $k_BT/2$ to the total internal energy of the system, allowing each mode to contribute k_BT to the internal energy. By calculating the power transferred to the transmission line using this expression for energy, we arrive at the result given in Equation 1.

More generally, for a circuit with a resistor of resistance R and a shunting capacitance C, circuit theory allows us to conclude that the Nyquist formula becomes

$$dV^2 = 4R_fk_BTdf \tag{2}$$

with

$$R_f = \frac{R}{1 + (2\pi fCR)^2} \tag{3}$$

2.2. Experimental Apparatus

A schematic of the experimental setup for Johnson noise measurement is shown in Figure 1. A resistor is attached to two alligator clips that protrude from the Johnson noise box, and a metal beaker is used to shield

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the resistor from external electromotive forces during the measurement. A switch connecting the circuit to an ohmmeter allows for a measurement of the resistance. During the Johnson noise measurement, this connection is switched off and the ohmmeter is disconnected to reduce external noise. Another switch shorts the circuit to the resistor, which allows us to distinguish between the measurement of the Johnson noise of our resistor and the intrinsic Johnson noise of the setup. The signal from the resistor is fed into an SR560 low noise preamp which serves as an amplifier of gain 500 and a high-pass filter with cutoff frequency of 1 kHz. The signal then enters a Krohn-Hite Model 3988 programmable filter, which serves as a low pass filter with cutoff at 50 kHz. The signal then enters the oscilloscope where we can measure the V_{rms}.

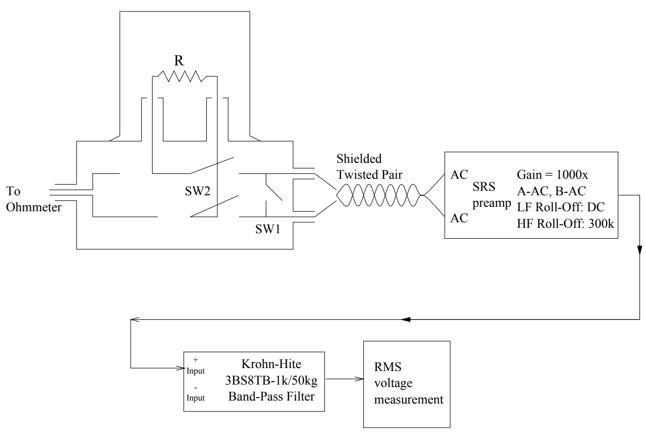

FIG. 1: Diagram of the apparatus for measuring Johnson noise. Image from [3].

In order to test the predictions of Nyquist's theory of Johnson noise, we need to know the gain as a function of frequency of our band-pass filter, as well as the shunting capacitance seen by the resistor. The gain curve is measured using a function generator to send signals of different frequencies through the measurement chain. The ratio of the input and output V_{rms} gives the gain at that frequency. Figure 2 shows the gain squared as a function of frequency. The measurement of the shunting capacitance is straightforward, and we found a value of $C = 22.7 \pm 1$ pF.

When taking measurements, the V_{rms} measured will be related to the V_{rms} produced by Johnson noise in the resistor and the gain $g(f)$ by

$$dV_{\text{meas}}^2 = [g(f)]^2 dV^2 \tag{4}$$

The orthogonality of each Fourier component of the signal allows us to integrate Equation 2 over the range of our band-pass to obtain an expression for the total mean squared voltage,

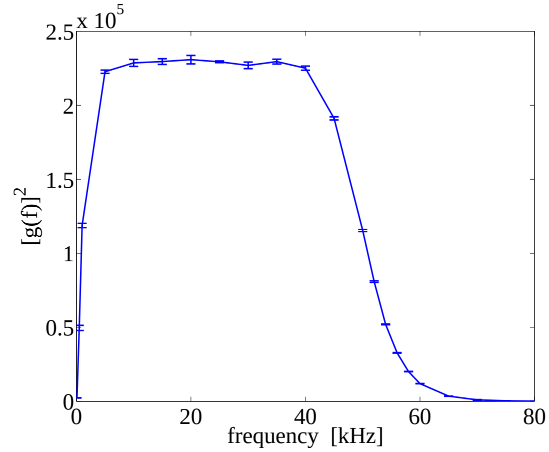

FIG. 2: Gain squared of the Johnson noise measurement chain, with steep dropoffs at 1 kHz and 50 kHz.

$$V^2 = 4Rk_BTG \tag{5}$$

where

$$G = \int_0^\infty \frac{[g(f)]^2}{1 + (2\pi fCR)^2} df \tag{6}$$

2.3. Measurements

Our investigation of Johnson noise consists of measuring the mean squared voltage for a variety of resistances, which will yield an estimate of Boltzmann's constant k_B. We also measure the dependence of the mean squared voltage as a function of temperature, which not only provides another estimate of k_B, but also gives the Celsius temperature of absolute zero.

2.3.1. Vary resistance

We took measurements of the Johnson noise for 9 different resistors at room temperature $T = 293$ K. To isolate the noise produced by the resistor, we alternated between measurements of with the resistor connected to the circuit, and the resistor shorted. The mean squared voltage due to the resistor is then given by $V_r^2 - V_s^2$. Five measurements were taken for each resistor to obtain an estimation of the random error of the measurement.

In order to calculate G for a given resistance, we performed a numerical integration of Equation 6 from 0.1 to 80 kHz. We used the trapezoidal method to calculate the integral, which, as shown by Bevington and Robinson [4], has a numerical error associated with each interval of $\frac{df}{12}h''(\xi)$, where h is the integrand, and ξ lies in the interval f and $f + df$. Using finite difference approximations of the second derivative, we found the numerical error to be 7 orders of magnitude less than the random errors of the measurement, so for our purposes we were able to treat the numerical integral as exact.

Our measurements allow us to solve Equation 5 for k_B for each resistor, and we obtained several values for Boltzmann's constant, each with an associated error. Taking a weighted mean that minimizes the residuals over all the resistances, we found a value of $k_B = 1.21 \pm 0.09 \times 10^{-23}$ m^2 kg s^{-2} T^{-1}. This is within 2σ of the accepted value of 1.38×10^{-23}.

2.3.2. *Vary temperature*

We also measured the dependence on temperature of the Johnson noise of a single resistor with resistance 817 kΩ. A low temperature measurement at T$= -196°$ C was made by submerging the resistor in a bath of liquid nitrogen. Other measurements were made by putting the resistor in a cylindrical oven, and monitoring the temperature using a mercury glass thermometer. Ten measurements were made at each temperature, again switching between the shorted circuit and the resistor. Calculating G as before, we made a plot of temperature versus $V^2/4RG$ (Figure 3). A linear fit to this data gives another estimate of k_B, as well as an estimate of the Celsius temperature of absolute zero, T_0. We found values of $k_B = 2.90 \pm 0.12 \times 10^{-23}$ and $T_0 = -245° \pm 12°$ C.

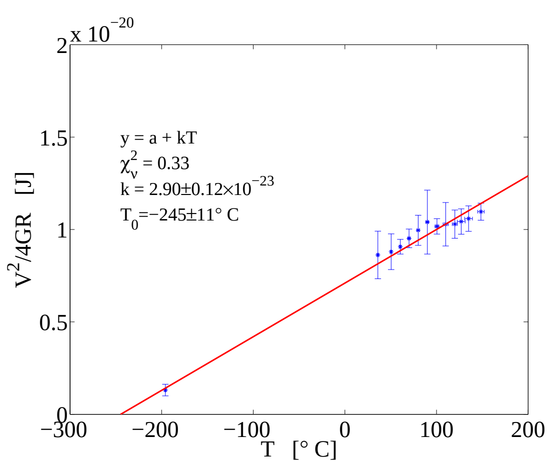

FIG. 3: Measurement of Johnson noise as a function of temperature. The linear fit shown in red has a slope of k_B and an x-intercept of T_0.

Our measurement of absolute zero was 3σ higher than the accepted value of -273, and the measurement of k_B is several standard deviations off. This indicates that there is likely some systematic error that we are not accounting for. One explanation for this error is external noise affecting our measurements, since this experiment is very sensitive to the presence of an external electromagnetic field.

3. SHOT NOISE

3.1. Theory

The noise due to the passage of a discrete charge carrier through the circuit is called shot noise. One example of where it might arise is the emission of photoelectrons in a circuit. The response of the circuit to a single photoelectron is to create an initial spike in the current which quickly settles back to the average current. Several of these events combined create noise in the circuitry which is related to the charge e the particle. Appendix C of [3] provides a derivation of the formula of the current created by shot noise, which is

$$d\langle I^2 \rangle = 2eI_{\text{ave}}df \tag{7}$$

Thus we see that the shot noise depends linearly on the charge e of the particle. Thus, a measurement of the shot noise of a system will allow us to calculate the charge of an electron.

3.2. Experimental Apparatus

The apparatus for measuring shot noise is shown in Figure 4. Within the photodiode box, a light bulb of adjustable intensity shines on the photodiode, which creates a current of photoelectrons. The noise measured in this current will be dominated by shot noise. The average DC current is measured using a multimeter, and the signal from the shot noise is amplified within the photodiode box. It then passes through the same preamplifier and band-pass chain as in the Johnson noise measurement, with a gain set at 2000. The AC mean squared voltage is then measured at the oscilloscope.

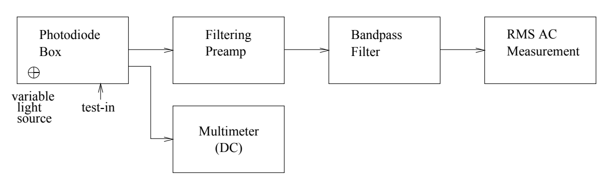

FIG. 4: Block diagram for the shot noise apparatus.

As with the Johnson noise measurement, we calibrated the measurement chain by finding the gain as a function of frequency of our measurement chain. Once again a function generator was used to input a test signal a known frequency into the photodiode box in order to compute the gain $g(f)$. The AC mean squared voltage will then be given by

$$V_0^2 = 2eI_{\text{ave}}R_f^2 \int_0^\infty [g(f)]^2\, df + V_A^2 \tag{8}$$

where $R_f = 475$ kΩ, and V_A^2 represents other contributions to noise in the system.

3.3. Measurement

By adjusting the current fed to the light bulb, we can change the intensity of incident light on the photodiode, and thus change the average DC current in the system. For 12 different values of I_{ave} we measured the root mean squared voltage of the AC current coming from the photodiode box. 10 measurements at each value of I_{ave} were taken in order to asses the random errors of the system.

Figure 5 shows a plot of the measured mean squared voltages. The x-axis gives the quantity $2R_f^2 G I_{\text{ave}}$, so that the slope of a line fitted to the data will be e, the charge of an electron. Here, G is the integral of the gain squared over the range of the band-pass filter.

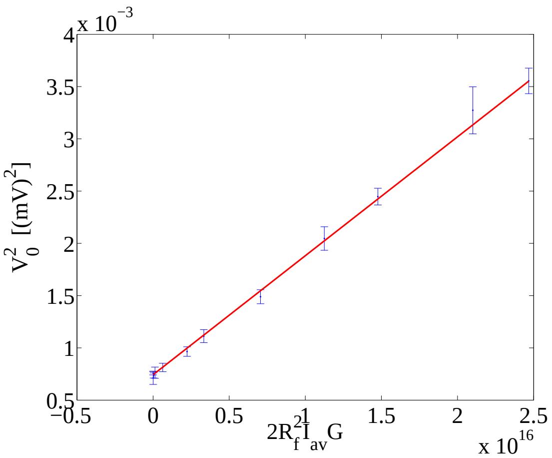

FIG. 5: Plot of shot noise from a photodiode. The linear fit shown in red has a slope of e.

The data yielded a value of $e = 1.138 \pm 0.032 \times 10^{-19}$ Coulombs, and the fit had a reduced chi squared of $\chi^2_\nu = 0.3$. This value for e differs from the accepted value of 1.602×10^{-19} by 30%. As with the Johnson noise measurement, it is likely external noise is interfering with the measurement chain to introduce a systematic error that is skewing our data.

4. CONCLUSIONS

In this experiment, we measured the two types of noise that are inherent in any circuit. We examined the dependence of Johnson noise on resistance, and using our measurements were able to measure Boltzmann's constant to be $k_B = 1.21 \pm 0.09 \times 10^{-23}$ m^2 kg s^{-2} T^{-1}, which is within two standard deviations of the accepted value. Our measurement of Johnson noise as function of temperature yielded gave $k_B = 2.90 \pm 0.12 \times 10^{-23}$ and the Celsius temperature of absolute zero $T_0 = -245° \pm 12°$ C, compared to the accepted value of $-273°$ C. Finally, our measurement of shot noise gave a measurement of the charge of an electron to be $e = 1.138 \pm 0.032 \times 10^{-19}$ Coulombs, which is 30% off the accepted value of 1.602×10^{-19}.

The fact that all our measurements were of the correct order of magnitude, but several standard deviations off the accepted values indicates that there was a systematic error present that we were not accounting for. This error likely arises from external electromagnetic fields, since our apparatus is surrounded by computers and other electronics. Also, the measurements made in this experiment are very sensitive to the configuration of the setup, and change the position of cables can have an effect on the measured noise. These effects combined may have caused our data to have been slightly skewed.

[1] J. B. Johnson, Physical Review **32** (1928).
[2] H. Nyquist, Physical Review **32** (1928).
[3] J. Lab Staff, *Johnson Noise and Shot Noise*, MIT Department of Physics (2010), lab guide.
[4] P. R. Bevington and D. K. Robinson, *Data Reduction and Error Analysis for the Physical Sciences* (McGraw-Hill, 2003), 3rd ed.

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